A Simple Finite Rotation Formulation for Composite Shell Elements

W. Wagner & F. Gruttmann

Institut für Baumechanik und Numerische Mechanik
Universität Hannover
Appelstr. 9A, D-3000 Hannover 1, FRG.

ABSTRACT

In this paper we derive a simple finite element formulation for geometrical nonlinear shell structures. The formulation bases on a direct introduction of the isoparametric finite element formulation into the shell equations. The element allows the occurrence of finite rotations which are described by two independent angles. A layerwise linear elastic material model for composites has been chosen. A consistent linearization of all equations has been derived for the application of a pure Newton method in the nonlinear solution process. Thus a quadratic convergence behaviour can be achieved in the vicinity of the solution point. Examples show the applicability and effectivity of the developed element.

KEY WORDS Nonlinear Shell Formulation Finite Rotations Composite Material

INTRODUCTION

In this paper we like to discuss the geometrical nonlinear behaviour of composite shell structures in the presence of finite rotations within the finite element method.

The application of composite materials became very popular in the last decades, especially in aircraft industries. The advantages of these materials are high strength and stiffness ratios coupled with a low specific weight. Thus, composites are used in highly loaded light weight structures. Often the designed constructions are thin shells which are very sensitive against loss of stability. Therefore the discussion of the stability behaviour is crucial for composite shell problems besides the description of material phenomena like matrix and fiber cracking or delamination. Due to these problems we have high requirements on the accuracy of the numerical calculations.

For this purpose we derive a general shell element which includes the geometrical nonlinear effects, especially finite rotations in an exact manner. Up to now a number of general finite shell elements including finite rotations are known. We mention the formulations of Ramm, Hughes et.al., Simo et.al., Stanley et.al., Gruttmann, Stein, Wriggers, Wriggers, Gruttmann, Parisch, Gebhardt, Schoop, Stander et.al., Başar & Ding, Dorninger & Rammerstorfer among others.

The derived element formulation is based on a Total Lagrangian formulation with Green-Lagrangian strains which are defined only by the deformed and undeformed base vectors of the shell mid–surface. The displacements are introduced with respect to the Cartesian base system. In this case a straightforward isoparametric formulation of the displacement field is possible, see e.g. Simo et.al., Wagner & Stein. Hence, no specific formulations like e.g. Christoffel tensors – as in classical shell theories – are necessary.

The description of the deformed and undeformed director is given with respect to the Cartesian coordinate system via two angles, originally introduced by Ramm\textsuperscript{3,4} for a degenerated shell element, here used in a different way. All necessary matrices and vectors are calculated only by standard linearization procedures acting on the base vectors and the directors. Due to this fact the curvatures of the initial configuration can be treated too. A simple formulation comes out for the consistent tangent stiffness matrix and the residual used in the nonlinear finite element analysis. This approach for the director vector leads to the same numerical results like an exponential update of the rotation tensor, see e.g. Simo et.al.\textsuperscript{1}.

In contrast to a degenerated formulation the weak form and the material law are formulated in terms of stress resultants on the shell mid–surface together with an analytical integration in thickness direction.

The composite material description is given with respect to local material axis and a transformation to global directions is given on element level. Standard procedures are used to modify the general 3D-material law for the two-dimensional case on the shell mid–surface.

Examples show the applicability of the proposed element for geometrical highly nonlinear shell structures of composite material.

The contents of the paper may be outlined as follows: The second section shows the basic kinematic assumptions whereas in the third section the material law for a composite shell formulation is discussed. Based on the weak form in the fourth section, the associated finite element formulation is given in section 5. Finally we show some illustrative examples in the last section.

KINEMATICS

The mathematical description of a point $P \in \mathcal{B}$ in shell space is based on the introduction of the position vector $\mathbf{p}$, which is a function of the convected coordinates $\Theta^i = \xi, \eta, \zeta$, see Figure 1. We have to distinguish between quantities in the reference configuration and the current configuration (marked by an upper bar). Thus the position of the points $\bar{P} \in \bar{\mathcal{B}}$ and $P \in \mathcal{B}$ is given by

$$
\mathbf{p} = \mathbf{x} + \zeta \mathbf{a}_3, \quad \bar{\mathbf{p}} = \bar{\mathbf{x}} + \zeta \bar{\mathbf{d}}, \quad -\frac{h}{2} \leq \zeta \leq +\frac{h}{2}.
$$

(1)

In (1) $\bar{\mathbf{x}}$ is the position vector to the shell mid–surface $\zeta = 0$ and $\bar{\mathbf{d}}$ a director vector which is fundamental to characterize the rotational behaviour. $\mathbf{d}$ is in general not perpendicular to the deformed mid–surface if we introduce a so called Reissner–Mindlin theory. In the special case that $\mathbf{d}$ coincide with the normal vector $\bar{\mathbf{a}}_3$, this kinematic assumption leads to the Kirchoff–Love theory which neglects transverse shear deformations.

The following quantities are necessary to describe the geometry of the shell in the reference configuration.
Figure 1 Kinematic of a thin shell

base vectors \( a_i = \{a_\alpha, a_3\} \):
\[
\begin{align*}
\alpha & = x_\alpha \\
3 & = a_1 \times a_2 / \|a_1 \times a_2\|
\end{align*}
\]

metric tensor:
\[
A = a_\alpha \otimes a^\alpha
\]

curvature tensor:
\[
B = -a_3 \otimes a^\alpha
\]

unit tensor:
\[
1 = a_\alpha \otimes a^\alpha + a_3 \otimes a_3
\]

base vectors in \( B \) \( g_i = \{g_\alpha, g_3\} \):
\[
\begin{align*}
\alpha & = a_\alpha + \zeta a_3 \\
3 & = a_3
\end{align*}
\]

The associated kinematic measures in the current configuration are defined in a similar way. In addition to the convected basis \( g_i \) the reciprocal basis \( g^i \) is given by the standard relation \( g_i \cdot g^k = \delta_i^k \). For the relation between base vectors in shell space \( B \) and on the shell mid-surface \( M \) we introduce the shifter tensor \( Z \)
\[
g_i = Za_i \quad g^i = Z^{-1} a_i
\]

with the standard definition of the shifter tensor
\[
Z = 1 - \zeta B.
\]

For the calculation of strain measures we make use of the deformation gradient \( F \) which is defined in convected coordinates by
\[
F = \frac{\partial p}{\partial \theta_k} \otimes g^k = g_k \otimes g^k = F_{ik} g^i \otimes g^k,
\]

with the components of \( F \)
\[
F_{ik} = g_i \cdot g_k.
\]

The strain measures for our shell problem can now be obtained from the three-dimensional theory using the Green–Lagrangian strain tensor \( E_B = \frac{1}{2}(F^T F - 1) \). With eq. (5) \( E_B \) can be stated in terms of the base vectors
\[
E_B = E_{ik} g^i \otimes g^k = \frac{1}{2} (g_i \cdot g_k - g_i \cdot g_k) g^i \otimes g^k.
\]
With the relation $E_B = Z^{-1} E Z^{-1}$ the associated Green–Lagrangian strain tensor on the shell mid–surface is given by

$$E = E_{ik} a^i \otimes a^k = \frac{1}{2} (g_i \cdot g_k - g_i \cdot g_k) a^i \otimes a^k.$$  \hspace{1cm} (8)

Up to now basic equations of a standard approach for a shell theory are derived. For a detailed discussion we refer to standard text books on shell theories.

Analyzing eq. (8) shows that we have to specify the base vectors in the reference and the current configuration which can be done using eqs. (2).

Furthermore we introduce explicitly the base vectors in $B$ from $\bar{p} = \bar{x} + \zeta \bar{d}$

$$g_\alpha = \bar{a}_\alpha + \zeta d_{1,\alpha} \hspace{1cm} g_3 = \bar{d}.$$  \hspace{1cm} (9)

The next step in a classical approach for the derivation of a shell theory is the description of the convected basis $\bar{g}_\alpha = \bar{a}_\alpha + \zeta d_{1,\alpha}$ in terms of the base vectors of the reference configuration and the displacement vector $u = u^\alpha a_\alpha + u^3 a_3$ (this means a description of $u$ with respect to the convected basis). It holds

$$\bar{g}_\alpha = \bar{a}_\alpha + \zeta d_{1,\alpha} = (a_\alpha + u_\alpha) + \zeta (a_{3,\alpha} + w_{1,\alpha}),$$  \hspace{1cm} (10)

with the difference vector $w = \bar{d} - a_3$, see e.g. Pietraszkiewicz 7. For the further derivation of a shell formulation the gradients $u_{1,\alpha}$ and $w_{1,\alpha}$ have to be specified. Here a number of complicated terms occur due to the definitions of $u$ and $w$ with respect to $a_\alpha$. Especially for the last term approximations are used due to the degree of rotation which then restrict the applicability of the introduced shell formulation. Again we refer to standard textbooks on shell theories for a detailed discussion of this problem.

Here we want to choose a different approach, where we will approximate the deformed base vectors $\bar{a}_\alpha$ directly. The main idea is the direct description of the convected basis $g_\alpha = \bar{a}_\alpha + \zeta d_{1,\alpha}$ without using the second relation defined in eq. (10).

From (8) we get for the components $E_{ik}$ of the Green–Lagrangian strains on the shell mid–surface which lead to the following strain measures

$$\epsilon_{\alpha\beta} = \frac{1}{2} [\bar{a}_\alpha \cdot \bar{a}_\beta - a_\alpha \cdot a_\beta]$$

$$\gamma_{\alpha 3} = \frac{1}{2} [\bar{a}_\alpha \cdot \bar{d}]$$

$$\kappa_{\alpha\beta} = \frac{1}{2} [\bar{a}_\alpha \cdot d_{1,\beta} + a_\beta \cdot d_{1,\alpha} - a_\alpha \cdot a_{3,\beta} - a_\beta \cdot a_{3,\alpha}].$$  \hspace{1cm} (11)

Here $\epsilon_{\alpha\beta}$ are the membrane strains, $\gamma_{\alpha 3}$ the shear strains and $\kappa_{\alpha\beta}$ the bending strains. $a_\alpha, a_3$ and $\bar{a}_\alpha, \bar{d}$ are the base vectors in the undeformed and deformed configurations, respectively. The indices $\alpha$ and $\beta$ range from 1 to 2.

Remark 1:

1) According to standard shell theories all terms associated with $\zeta^2$ are neglected. This assumption is true for thin shells.
2) It occurs no shear flexure term which is consistent to the introduced kinematic assumption. \((\mathbf{d}_\alpha \cdot \mathbf{d} = 0)\).

3) The construction of \(\mathbf{a}_3\) leads to an unit normal vector. The director vector \(\mathbf{d}\) is obtained by a pure rotation of \(\mathbf{a}_3\) which will be shown later. Thus \(\mathbf{d}\) is an unit vector and within this theory no change of thickness \(h\) is described.

A detailed description of the calculation of the base vectors \(\mathbf{a}_\alpha, \mathbf{a}_3\) and \(\bar{\mathbf{a}}_\alpha\) is given within the isoparametric finite element formulation.

The definition of the director vector \(\mathbf{d}\) bases on the introduction of two independent rotation angles. This strategy has been used by Ramm\(^{3,4}\) for a degenerated shell element. Ramm introduces for the displacement vector

\[
\mathbf{v} = \mathbf{u} + \zeta (\mathbf{d} - \mathbf{a}_3),
\]

\[
\begin{align*}
\mathbf{v} &= \mathbf{u} + \zeta \begin{pmatrix} 
\cos \bar{\psi}_1 - \cos \psi_{01} \\
\cos \bar{\psi}_2 - \cos \psi_{02} \\
\cos \bar{\psi}_3 - \cos \psi_{03}
\end{pmatrix}
\end{align*}
\]

where \(\bar{\psi}_i\) are the angles between the director vector \(\mathbf{d}\) and the Cartesian coordinate axes \(x_i\). Similar the angles \(\psi_{0i}\) are the angles between the undeformed normal vector \(\mathbf{a}_3\) and the Cartesian coordinate axes \(x_i\) which describe the undeformed state. A further description is possible with two independent rotation angles. Here we choose the angles \(\beta_1\) and \(\beta_2\), see Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Description of the director vector \(\mathbf{d}\) in shellspace}
\end{figure}

We define the angles in the deformed configuration \(\bar{\beta}_1\) and \(\bar{\beta}_2\) by

\[
\begin{align*}
\bar{\beta}_1 &= \beta_{01} + \beta_1 \\
\bar{\beta}_2 &= \beta_{02} + \beta_2
\end{align*}
\]

with the angles \(\beta_{01}\) and \(\beta_{02}\) of the undeformed reference configuration. Thus the director vector in the deformed configuration is given with respect to the independent angles by

\[
\mathbf{d} = \begin{pmatrix} 
\cos \bar{\beta}_1 \sin \bar{\beta}_2 \\
\sin \bar{\beta}_1 \sin \bar{\beta}_2 \\
\cos \bar{\beta}_2
\end{pmatrix}
\]

With this choice of independent rotation variables all positions of the undeformed director can be described, except the situations where the director vector coincides with the \(x_3\)-axis.
\( \beta_{02} = 0^0, \beta_{01} \text{ is undefined} \). Ramm\(^3,4\) propose to use a local modified coordinate system as a possibility to overcome this problem. Furthermore – due to the problem – it is possible to define an associated value of the angle \( \beta_{01} \).

Example: One of the standard test examples for shell elements is a spherical shell with a single load in \( x_3 \)-direction, see e.g. McNeal, Harder\(^15\), where we have the above mentioned situation at the top of the shell with \([x_1, x_2, x_3] = [0, 0, R]\). Here, \( \beta_{01} = 45^0 \) leads to a correct solution, see also our first example (clamped rubber shell) illustrated in Fig. 4.

**STRESSES AND MATERIAL LAW**

In this section we like to discuss the chosen material law. Here we use a layerwise description of the material behaviour which is valid for composite materials. We assume a purely elastic behaviour without damage or crack effects. Then the material law for the composite material bases on a linear relation between the 2\textsuperscript{nd}–Piola–Kirchoff stresses \( \sigma, \tau \) and the components of the Green–Lagrangian strain tensor, which is valid for small strains. A two dimensional material law in the tangent plane is then formulated for a layer \( k \) ( \( k = 1, N\text{LAY} \), where \( N\text{LAY} \) is the total number of layers, from

\[
\sigma^k = C^k_G [\epsilon + \zeta^k \kappa], \quad \tau^k = \tilde{C}^k_G \gamma. \tag{15}
\]

Here the main components of the stress and strain tensors are summarized in vectors \( [\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T, \tau = [\tau_{13}, \tau_{13}]^T, \epsilon = [\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}]^T, \kappa = [\kappa_{11}, \kappa_{22}, 2\kappa_{12}]^T, \gamma = [\gamma_{13}, \gamma_{13}]^T] \). Stresses and strains are given in global directions. \( \zeta^k \) describes the coordinate in thickness direction of the shell. The matrices \( C^k_G \) and \( \tilde{C}^k_G \) are calculated from the transformation of the locally defined orthotropic material matrices for shell like structures with the elasticity matrices

\[
C^k_L = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix}_L, \quad \tilde{C}^k_L = \begin{bmatrix}
\tilde{C}_{11} & 0 \\
0 & \tilde{C}_{22}
\end{bmatrix}_L \tag{16}
\]

in the globally introduced coordinate system.

The transformation in the tangent plane between a local \( (x_1L, x_2L) \) coordinate system and a global \( (x_1G, x_2G) \) coordinate system is given by

\[
\begin{bmatrix}
x_1L \\
x_2L
\end{bmatrix} = \begin{bmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{bmatrix} \cdot \begin{bmatrix}
x_1G \\
x_2G
\end{bmatrix} \quad x_L = T_x x_G \tag{17}
\]

with the rotation angle \( \varphi \). The derivation of the strains bases on the chain rule. For example we get for

\[
\epsilon_{11L} = \frac{\partial u_{1L}}{\partial x_{1L}} = \frac{\partial u_{1L}}{\partial x_{1G}} \cdot \frac{\partial x_{1G}}{\partial x_{1L}} + \frac{\partial u_{1L}}{\partial x_{2L}} \cdot \frac{\partial x_{2L}}{\partial x_{1L}}, \tag{18}
\]

Introducing (17)

\[
\begin{align*}
u_{1L} &= \cos \varphi u_{1G} + \sin \varphi u_{2G} \quad \text{in} \quad \frac{\partial u_{1L}}{\partial x_{1G}} \quad \text{in} \quad \frac{\partial x_{1G}}{\partial x_{1L}} \\
x_{1G} &= \cos \varphi x_{1L} - \sin \varphi x_{2L} \quad \text{in} \quad \frac{\partial x_{1G}}{\partial x_{1L}}
\end{align*}
\]
leads to the result
\[ \epsilon_{11L} = \cos^2 \varphi \epsilon_{11G} + \sin^2 \varphi \epsilon_{22G} + \sin \varphi \cos \varphi \epsilon_{12G}, \]

Thus the following relation can be derived
\[ \epsilon_L = T_\epsilon \epsilon_G \] (19)

with
\[ T_\epsilon = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2 - s^2 \end{bmatrix} \]
\[ s = \sin \varphi \]
\[ c = \cos \varphi. \]

From a comparison of the specific internal energy
\[ dW = \frac{1}{2} \sigma_L^T \epsilon_L = \frac{1}{2} \sigma_G^T \epsilon_G \] (20)

we get the transformation for the stresses
\[ \sigma_L = T_\sigma \sigma_G \quad \text{with} \quad T_\sigma = T_\epsilon^{T-1}, \] (21)

and finally the relation for the material law
\[ \sigma_G = C_G \epsilon_G \quad \sigma_L = C_L \epsilon_L \quad \text{with} \quad C_G = T_\epsilon^T C_L T_\epsilon. \] (22)

This transformation is valid for the membrane strains \( \epsilon \) and the bending strains \( \kappa. \)

Similar the transformation of the shear stresses and strains has to be carried out. With \( T_\gamma = T_x \) it holds
\[ \tilde{C}_G = T_x^T \tilde{C}_L T_x. \] (23)

After introducing this relation the following global material parameter occur
\[
\begin{align*}
C_{11G} &= c^4 C_{11L} + 2s^2 c^2 (C_{12L} + 2C_{33L}) + s^4 C_{22L} \\
C_{22G} &= s^4 C_{11L} + 2s^2 c^2 (C_{12L} + 2C_{33L}) + c^4 C_{22L} \\
C_{12G} &= s^2 c^2 (C_{11L} + C_{22L} - 4C_{33L}) + (s^4 + c^4)C_{12L} \\
C_{13G} &= c^3 s(C_{11L} - C_{12L} - 2C_{33L}) + s^3 c(C_{12L} - C_{22L} + 2C_{33L}) \\
C_{23G} &= s^3 c(C_{11L} - C_{12L} - 2C_{33L}) + c^3 s(C_{12L} - C_{22L} + 2C_{33L}) \\
C_{33G} &= s^2 c^2 (C_{11L} + C_{22L} - 2C_{12L} - 2C_{33L}) + (s^4 + c^4)C_{33L} \\
\end{align*}
\]
\[
\begin{align*}
\tilde{C}_{11G} &= c^2 \tilde{C}_{11L} + s^2 \tilde{C}_{22L} \\
\tilde{C}_{22G} &= s^2 \tilde{C}_{11L} + c^2 \tilde{C}_{22L} \\
\tilde{C}_{12G} &= sc(\tilde{C}_{11L} - \tilde{C}_{22L}) \\
\end{align*}
\]

Within the transformation the symmetry of the material matrix is preserved, but the matrices are now fully populated.
A detailed derivation of these relations can be found in e.g. Tsai 5.

The elements of the material matrices of a layer \( k \) depend on the elastic modules \( E_i \) of a three-dimensional material law in the following way, see e.g. Tsai 5,

\[
C_{11L} = \frac{1}{(1 - \nu^2 E_2/E_1)} E_1 \\
C_{22L} = \frac{1}{(1 - \nu^2 E_2/E_1)} E_2 \\
C_{12L} = \frac{1}{(1 - \nu^2 E_2/E_1)} E_2 \nu \\
C_{33L} = \tilde{C}_{11L} = G_{12} \\
\tilde{C}_{22L} = G_{23}
\]  

(25)

Axis 1 is parallel to the fibers of the considered layer, while axis 2 is normal to the fiber direction.

The shell stress resultants and stress couples are introduced in the common way. With respect to the numerical formulation we have not to distinguish between co– and contravariant components of the stress resultants and couples.

The components of the associated tensors are defined by

\[
N_{\alpha\beta} := \int_{(h)} \sigma_{\alpha\beta} \det Z \, d\zeta , \\
M_{\alpha\beta} := \int_{(h)} \zeta \sigma_{\alpha\beta} \det Z \, d\zeta , \\
Q_\alpha := \int_{(h)} \kappa \tau_\alpha \det Z \, d\zeta ,
\]

(26)

and summarized in the vectors \( \mathbf{N} = [N_{11}, N_{22}, N_{12}] \), \( \mathbf{M} = [M_{11}, M_{22}, M_{12}] \), \( \mathbf{Q} = [Q_{13}, Q_{23}] \). Here, \( \kappa \) is the shear correction factor usually chosen as \( 5/6 \).

Usually the determinant of the shifter tensor

\[
\det Z = \det (1 - \zeta \mathbf{B}) = 1 - \zeta (B_1^1 + B_2^2) + \zeta^2 (B_1^1 B_2^2 - B_1^2 B_2^1)
\]

(27)

is approximated by the first term which is valid for thin shells.

Within the material law for these averaged global stress resultants we can see the well known coupling effect between membrane and bending terms

\[
\begin{bmatrix}
\mathbf{N} \\
\mathbf{M} \\
\mathbf{Q}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{D}^m & \mathbf{D}^{mb} & 0 \\
\mathbf{D}^{mbT} & \mathbf{D}^b & 0 \\
0 & 0 & \mathbf{D}^s
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathbf{\epsilon} \\
\mathbf{\kappa} \\
\mathbf{\gamma}
\end{bmatrix} 
\]

(27)
with

\[
D^m = \sum_{k=1}^{NLAY} C^k h^k \\
D^b = \sum_{k=1}^{NLAY} C^k \left( \frac{h^k}{12} h^k \zeta_s^2 \right) \\
D^{mb} = \sum_{k=1}^{NLAY} C^k h^k \zeta_s^k \\
D^s = \sum_{k=1}^{NLAY} \tilde{C}^k h^k. 
\]  

(28)

In (28) \( h^k \) is the thickness of the \( k \)-th layer, \( \zeta_s^k \) is the distance from the midpoint of the considered layer to the reference surface and \( NLAY \) is the total number of layers.

**WEAK FORM – PRINCIPLE OF VIRTUAL WORK**

The numerical treatment within the finite element method is based on the principal of virtual work. It is given with reference to the undeformed shell configuration in a Total–Lagrangian description by

\[
D \pi \cdot \delta v = \int_\Omega \left[ N \cdot \delta \epsilon + M \cdot \delta \kappa + Q \cdot \delta \gamma \right] d\Omega - \int_\Omega \rho \hat{b} \cdot \delta v \, d\Omega - \int_{\Omega_v} \hat{t} \cdot \delta v \, d\Omega. 
\]  

(29)

In eq. (29) \( \pi \) is the potential energy calculated in the domain \( \Omega \), the first r.h.s. term describes the virtual work of the internal forces while the last term describes the virtual work of the external forces, given in a simplified formulation. The geometrical nonlinear load deflection behaviour is calculated with a Newton–type iteration procedure. For this purpose we need the consistent linearization of the principle of virtual work

\[
D \left[ D \pi \cdot \delta v \right] \cdot \Delta v = \int_\Omega \delta \epsilon \cdot D^m \Delta \epsilon \, d\Omega + \int_\Omega \delta \kappa \cdot D^{mb} \Delta \kappa \, d\Omega \\
+ \int_\Omega \delta \kappa \cdot D^{mbT} \Delta \epsilon \, d\Omega + \int_\Omega \delta \kappa \cdot D^b \Delta \kappa \, d\Omega \\
+ \int_\Omega \delta \gamma \cdot D^s \Delta \gamma \, d\Omega + \int_\Omega \delta \Delta \epsilon \cdot N \, d\Omega \\
+ \int_\Omega \delta \Delta \kappa \cdot M \, d\Omega + \int_\Omega \delta \Delta \gamma \cdot Q \, d\Omega. 
\]  

(30)

The linearizations and variations of the strain measures are shown in the next section.
FINITE ELEMENT FORMULATION

In this section we will discuss briefly the formulation of the associated finite element. We introduce a general finite element discretization

$$B^h = \bigcup_{e=1}^{n_e} \Omega_e$$

with \( n_e \) elements. The formulation is based on the isoparametric concept.

Within a single element \( \Omega_e \) the position vector \( \mathbf{x} \) and the displacement vector \( \mathbf{u} \) are approximated by

$$\mathbf{x}^h = \{x_1, x_2, x_3\}^T = \sum_{I=1}^{nel} N_I \mathbf{x}_I ,$$

$$\mathbf{u}^h = \{u_1, u_2, u_3\}^T = \sum_{I=1}^{nel} N_I \mathbf{u}_I .$$

(31)

\( N_I \) are the shape functions and \( \mathbf{x}_I , \mathbf{u}_I \) are the vector of nodal coordinates and the nodal displacement vector. Note that the displacement vector is introduced with respect to the Cartesian coordinate system. The rotation angles \( \beta_{0\alpha} \) and \( \beta_\alpha \) are approximated similar

$$\beta_0^h = \{\beta_{01}, \beta_{02}\}^T = \sum_{I=1}^{nel} N_I \beta_{0I} ,$$

$$\beta^h = \{\beta_1, \beta_2\}^T = \sum_{I=1}^{nel} N_I \beta_I .$$

(32)

We introduce as shape functions \( N_I \) Lagrange- or Serendipity-shape functions. Here \( nel \) defines the total number of nodes at the element. As an example, it holds for the four node element

$$N_I(\xi, \eta) = \frac{1}{4} (1 + \xi \xi_I) (1 + \eta \eta_I) .$$

(33)

The variation of the strain measures - used in eq.(29),(30) are based on the variation of the base vectors. The undeformed basis is defined at the Gauss points from the local base vectors \( \mathbf{a}_\xi \) and \( \mathbf{a}_\eta \)

$$\mathbf{a}_\xi = \sum_{I=1}^{nel} N_{I,\xi} \mathbf{x}_I ,$$

$$\mathbf{a}_\eta = \sum_{I=1}^{nel} N_{I,\eta} \mathbf{x}_I ,$$

(34)

$$\mathbf{a}_\zeta = (\mathbf{a}_\xi \times \mathbf{a}_\eta) / \| \mathbf{a}_\xi \times \mathbf{a}_\eta \| ,$$

via

$$\mathbf{a}_1 = \mathbf{a}_\xi / \| \mathbf{a}_\xi \| ,$$

$$\mathbf{a}_3 = \mathbf{a}_\zeta ,$$

$$\mathbf{a}_2 = \mathbf{a}_3 \times \mathbf{a}_1 .$$

(35)
From eq. (35) we use only the tangential vectors $\mathbf{a}_1, \mathbf{a}_2$. As normal vector we employ the base vector $\mathbf{a}_3$ with respect to the rotation angles $\beta_{01}, \beta_{02}$, see eq. (14)

$$\mathbf{a}_3 = \begin{bmatrix} \cos \beta_{01} \sin \beta_{02} \\ \sin \beta_{01} \sin \beta_{02} \\ \cos \beta_{02} \end{bmatrix}.$$ (36)

This strategy is consistent with the formulation for the deformed director vector $\mathbf{d}$.

At the beginning the angles $\beta_{01}, \beta_{02}$ have to be defined. This can be done via different ways. One way is to read the base vectors $\mathbf{a}_3$ at the nodes from an input file. Another possibility is the definition of a local basis at the nodes similar to the eqs. (34–35). This element dependent procedure leads to different normal vectors at one node with respect to the associated elements. This problem always happens within the discretization of curved structures. An averaging process may be necessary to improve the results, see e.g. Büchter (6). For special structures for example plate, cylinder, hypar, hyperboloid where the surface of the structure can be described analytically, $\mathbf{a}_3$ can be introduced at the nodes exactly via the surface function.

In order to introduce the deformed basis vectors it is necessary to define the derivatives with respect to the local Cartesian base system $\theta_\alpha$. Within the finite element concept this will be done at the Gauss points. It holds for the derivatives of the position vector

$$\frac{\partial \mathbf{x}}{\partial \xi_\alpha} = \frac{\partial \mathbf{x}}{\partial \theta_\beta} \frac{\partial \theta_\beta}{\partial \xi_\alpha} \quad \text{with } \xi_1 = \xi, \xi_2 = \eta.$$ (37)

With (2) and (34) we can introduce the base vectors $\mathbf{a}_\xi_\alpha$ and $\mathbf{a}_\beta$ by

$$\mathbf{a}_\xi_\alpha = \mathbf{a}_\beta \frac{\partial \theta_\beta}{\partial \xi_\alpha}$$

which leads with $\mathbf{a}_\xi_\alpha \cdot \mathbf{a}_\beta = \frac{\partial \theta_\beta}{\partial \xi_\alpha}$ to the components $J_{\alpha\beta}$ of the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{a}_\xi \cdot \mathbf{a}_1 & \mathbf{a}_\xi \cdot \mathbf{a}_2 \\ \mathbf{a}_\eta \cdot \mathbf{a}_1 & \mathbf{a}_\eta \cdot \mathbf{a}_2 \end{bmatrix}.$$ (38)

Thus, the derivatives $\frac{\partial \mathbf{x}}{\partial \xi_\alpha}$ are defined by

$$\frac{\partial \mathbf{x}}{\partial \xi_\alpha} = J_{\alpha\beta} \frac{\partial \mathbf{x}}{\partial \theta_\beta}.$$ (39)

The same formulation holds for the shape functions $N_I$. The derivatives of $N_I$ with respect to $\theta_\alpha$ are then derived from the inverse relation

$$\begin{bmatrix} N_{I,1} \\ N_{I,2} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} N_{I,\xi} \\ N_{I,\eta} \end{bmatrix}.$$ (40)

With these derivations at hand the deformed base vectors $\bar{\mathbf{a}}_\alpha$ are given at the integration points by
The director vector $\mathbf{d}$ has been defined in eq. (14). The calculation of $\mathbf{d}$ at the Gauss points bases on the interpolation of the rotation angles $\beta_{0,\alpha}$ and $\beta_{1,\alpha}$ defined in eq. (32).

The derivation of $\mathbf{d}$ with respect to $\alpha$ is defined by

$$
\mathbf{d}_{1,\alpha} = \mathbf{d}_{1,\beta_1} \beta_{1,\alpha} + \mathbf{d}_{2,\beta_2} \beta_{2,\alpha}
$$

with

$$
\mathbf{d}_1 = \begin{bmatrix}
-\sin \beta_1 \sin \beta_2 \\
\cos \beta_1 \sin \beta_2 \\
0
\end{bmatrix},
\mathbf{d}_2 = \begin{bmatrix}
\cos \beta_1 \cos \beta_2 \\
\sin \beta_1 \cos \beta_2 \\
-\sin \beta_2
\end{bmatrix}.
$$

With a similar definition for $\mathbf{a}_{3,\alpha}$ all base vectors and derivations are given for the formulation of the strain measures defined in eq. (11).

Within the principle of virtual work the variations of the strain measures are necessary.

Based on the variation of the membrane strains

$$
\delta \epsilon_{\alpha \beta} = \frac{1}{2} [\delta \mathbf{a}_{\alpha} \cdot \mathbf{a}_{\beta} + \mathbf{a}_{\alpha} \cdot \delta \mathbf{a}_{\beta}]
$$

and the variation of the deformed base vectors $\mathbf{a}_{\alpha}$

$$
\delta \mathbf{a}_{\alpha} = \sum_{I=1}^{nel} N_{I,\alpha} \delta \mathbf{u}_I
$$

we create a finite element formulation with the $\mathbf{B}$-matrices

$$
\delta \mathbf{e}^b = \sum_{I=1}^{nel} \mathbf{B}_I^m \delta \mathbf{u}_I
$$

with the $\mathbf{B}$-matrix of the membrane parts at node $I$

$$
\mathbf{B}_I^m = \begin{bmatrix}
\mathbf{a}_1^T N_{I,1} \\
\mathbf{a}_2^T N_{I,2} \\
(\mathbf{a}_1^T N_{I,2} + \mathbf{a}_2^T N_{I,1})
\end{bmatrix}
$$

The variation of the shear strains is given by

$$
\delta \gamma_{\alpha} = \delta \mathbf{a}_{\alpha} \cdot \mathbf{d} + \mathbf{a}_{\alpha} \cdot \delta \mathbf{d}.
$$

In addition we need the variation of $\mathbf{d}$

$$
\delta \mathbf{d} = \mathbf{d}_{1,\beta_1} \delta \beta_1 + \mathbf{d}_{2,\beta_2} \delta \beta_2 = \tilde{\mathbf{d}} \delta \beta.
$$
The finite element formulation of these strains is

$$
\delta \sigma^h = \sum_{I=1}^{nel} B_I^s \begin{bmatrix}
\delta u_I \\
\delta \beta_{1I} \\
\delta \beta_{2I}
\end{bmatrix}.
$$

(50)

with the \(B\)-Matrix of the shear terms at node \(I\)

$$
B_I^s = \begin{bmatrix}
d_T N_{I,1} & \bar{a}_1^T d_1 N_I & \bar{a}_2^T d_2 N_I \\
d_T N_{I,2} & \bar{a}_1^T d_1 N_I & \bar{a}_2^T d_2 N_I
\end{bmatrix}
$$

(51)

The variation of the bending part is given by

$$
\delta \kappa_{\alpha\beta} = \delta a_{\alpha} \cdot d_{\beta} + a_{\alpha} \cdot \delta d_{\beta}.
$$

For this purpose we have to introduce the derivation with respect to \(\theta_{\alpha}\) of the variation of the director vector \(d\)

$$
\delta d_{\alpha} = d_{\beta} \delta \beta_{\beta,\alpha} + d_{\gamma,\alpha} \delta \beta_{\gamma}
$$

(52)

with

$$
\delta d_{1,1} = d_1 \delta \beta_{1,1} + d_2 \delta \beta_{2,1} + d_{1,1} \delta \beta_1 + d_{2,1} \delta \beta_2 \\
\delta d_{1,2} = d_1 \delta \beta_{1,2} + d_2 \delta \beta_{2,2} + d_{1,2} \delta \beta_1 + d_{2,2} \delta \beta_2
$$

(53)

and

$$
\begin{align*}
d_{1,1} &= d_{11} \bar{\beta}_{1,1} + d_{12} \bar{\beta}_{2,1} \\
d_{1,2} &= d_{11} \bar{\beta}_{1,2} + d_{12} \bar{\beta}_{2,2} \\
d_{2,1} &= d_{21} \bar{\beta}_{1,1} + d_{22} \bar{\beta}_{2,1} \\
d_{2,2} &= d_{21} \bar{\beta}_{1,2} + d_{22} \bar{\beta}_{2,2}
\end{align*}
$$

(54)

and

$$
\begin{align*}
d_{11} &= \begin{bmatrix} -\cos \bar{\beta}_1 \sin \bar{\beta}_2 \\ -\sin \bar{\beta}_1 \sin \bar{\beta}_2 \\ 0 \end{bmatrix}, \\
d_{12} &= \begin{bmatrix} -\sin \bar{\beta}_1 \cos \bar{\beta}_2 \\ \cos \bar{\beta}_1 \cos \bar{\beta}_2 \\ 0 \end{bmatrix}, \\
d_{22} &= \begin{bmatrix} -\cos \bar{\beta}_1 \sin \bar{\beta}_2 \\ -\sin \bar{\beta}_1 \sin \bar{\beta}_2 \\ -\cos \bar{\beta}_2 \end{bmatrix}
\end{align*}
$$

(55)

Based on these matrices the variation of the bending part is given by

$$
\delta \kappa^b = \sum_{I=1}^{nel} B_I^b \begin{bmatrix}
\delta u_I \\
\delta \beta_{1I} \\
\delta \beta_{2I}
\end{bmatrix}
$$

(56)

with the \(B\)-Matrix of the bending part at node \(I\)

$$
B_I^b = \begin{bmatrix}
d_{1,1}^T N_{I,1} & \bar{a}_1^T (d_1 N_{I,1} + d_{1,1} N_I) & \bar{a}_2^T (d_2 N_{I,1} + d_{2,1} N_I) \\
d_{1,2}^T N_{I,1} & \bar{a}_1^T (d_1 N_{I,1} + d_{1,2} N_I) & \bar{a}_2^T (d_2 N_{I,1} + d_{2,2} N_I) \\
d_{1,1}^T N_{I,2} & \bar{a}_1^T (d_1 N_{I,2} + d_{1,1} N_I) & \bar{a}_2^T (d_2 N_{I,2} + d_{2,1} N_I) \\
+d_{1,2}^T N_{I,1} & +\bar{a}_2^T (d_1 N_{I,1} + d_{1,1} N_I) & +\bar{a}_2^T (d_2 N_{I,1} + d_{2,1} N_I)
\end{bmatrix}.
$$

(57)

Based on this element specific formulation we define the residual (on element level) in a standard way for a nodal displacement vector \(v_I = [u, \beta_1, \beta_2]^T\)
\[
G_e^h(\mathbf{v}_e, \delta \mathbf{v}_e) = \sum_{l=1}^{\text{nel}} \delta \mathbf{v}_l^T \left\{ \int_{\Omega_e} \mathbf{B}^m_l(\mathbf{v}_e)^T \mathbf{N}(\mathbf{v}_e) + \mathbf{B}^b_l(\mathbf{v}_e)^T \mathbf{M}(\mathbf{v}_e) \right. \\
\left. + \mathbf{B}^s_l(\mathbf{v}_e)^T \mathbf{Q}(\mathbf{v}_e) \right\} d\Omega_e - \int_{\partial \Omega_e} \mathbf{N}_l^T \delta \mathbf{d}(\partial \Omega) \right\}.
\]

The tangential stiffness matrix can be calculated from

\[
DG_e^h(\mathbf{v}_e, \delta \mathbf{v}_e) \Delta \mathbf{v}_e = \sum_{l=1}^{\text{nel}} \sum_{K=1}^{\text{nel}} \delta \mathbf{v}_l^T \left[ \int_{\Omega_e} \mathbf{B}_l^T(\mathbf{v}_e) \mathbf{D} \mathbf{B}_K(\mathbf{v}_e) \right] d\Omega_e + \int_{\Omega_e} \mathbf{G}_{IK} d\Omega \Delta \mathbf{v}_K.
\]

Within the 2\text{nd} term in eq.\,(59) – the geometrical matrix \( \mathbf{K}_\sigma \) – we need the linearizations of the virtual strains. After multiplication with the associated stress resultants , see eq. (26) we obtain the terms for \( \mathbf{G}_{IK} \) in \( \mathbf{K}_\sigma \). For example the membrane part is presented by

\[
\delta \mathbf{v}_l^T \mathbf{K}_\sigma(N)_e \Delta \mathbf{v}_e = \sum_{l=1}^{\text{nel}} \sum_{K=1}^{\text{nel}} \left[ \delta \mathbf{u}_l, \delta \beta_{11}, \delta \beta_{21} \right]^T \int_{\Omega_e} \mathbf{G}_{IK}(m) d\Omega \left[ \begin{array}{c} \Delta \mathbf{u}_K \\ \delta \beta_{1K} \\ \delta \beta_{2K} \end{array} \right].
\]

With

\[
\mathbf{G}_{IK}(m) = \begin{bmatrix} \mathbf{1}_{3x3} & \mathbf{S} & \mathbf{0}_{3x2} \\ \mathbf{0}_{2x3} & \mathbf{0}_{2x2} \end{bmatrix}
\]

and

\[
\mathbf{S} = \begin{bmatrix} N_{I,1} & N_{K,1} & N_{I,1} N_{K,1} + N_{I,2} N_{K,2} N_{22} + (N_{I,1} N_{K,2} + N_{I,2} N_{K,1}) N_{12} \end{bmatrix}.
\]

The matrices and stress resultants used are defined above. Due to the finite rotation formulation shear and bending part occur too. The bending part and the shear part of the geometrical matrix are calculated in a similar way.

For the shear part we need the variation of the linearization of \( \mathbf{\gamma} \)

\[
\delta \Delta \mathbf{\gamma}_\alpha = \delta \mathbf{a}_\alpha \cdot \Delta \mathbf{d} + \Delta \mathbf{a}_\alpha \cdot \delta \mathbf{d} + \mathbf{a}_\alpha \cdot \delta \Delta \mathbf{d}
\]

and the variation of the linearization of \( \mathbf{d} \) which is given by

\[
\delta \Delta \mathbf{d} = \mathbf{d}_{\alpha\beta} \delta \beta_2 \Delta \beta_\alpha = \mathbf{d}_{11} \delta \beta_1 \Delta \beta_1 + \mathbf{d}_{12} \delta \beta_2 \Delta \beta_1 + \mathbf{d}_{21} \delta \beta_1 \Delta \beta_2 + \mathbf{d}_{22} \delta \beta_2 \Delta \beta_2
\]

The part \( \mathbf{G}_{IK}(s) \) is then defined by

\[
\mathbf{G}_{IK}(s) = \begin{bmatrix} \mathbf{0}_{3x3} & \mathbf{d}_1^T \mathbf{Q}_\alpha N_{I,\alpha} N_{K} & \mathbf{d}_2^T \mathbf{Q}_\alpha N_{I,\alpha} N_{K} \\ \mathbf{d}_1^T \mathbf{Q}_\alpha N_{I,\alpha} N_{K} & \mathbf{a}_1^T \mathbf{d}_1^T \mathbf{Q}_\alpha N_{I} N_{K} & \mathbf{a}_1^T \mathbf{d}_2^T \mathbf{Q}_\alpha N_{I} N_{K} \\ \mathbf{d}_2^T \mathbf{Q}_\alpha N_{I,\alpha} N_{K} & \mathbf{a}_2^T \mathbf{d}_1^T \mathbf{Q}_\alpha N_{I} N_{K} & \mathbf{a}_2^T \mathbf{d}_2^T \mathbf{Q}_\alpha N_{I} N_{K} \end{bmatrix}.
\]
For the bending part we need the variation of the linearization of $\epsilon$

$$\delta \Delta \epsilon_{\alpha \beta} = \delta \bar{a}_\alpha \cdot \Delta d_{\alpha \beta} + \Delta \bar{a}_\alpha \cdot \delta d_{\alpha \beta} + \bar{a}_\alpha \cdot \delta \Delta d_{\alpha \beta}$$  (66)

and the variation of the linearization of $d_{\alpha \beta}$ which is given by

$$\delta \Delta d_{\alpha \beta} = d_{\alpha \gamma \delta \beta \delta \beta} \delta \beta \gamma \Delta \beta_\alpha + d_{\alpha \gamma \delta \beta} \Delta \beta_\alpha + \delta \beta_\gamma \Delta \beta_\alpha \beta$$  (67)

Here the vectors

$$d_{111} = d_{221} = d_{212} = d_{122} = -d_1$$

$$d_{222} = -d_2$$

$$d_{211} = d_{121} = -d_{112}$$  (68)

with

$$d_{112} = \begin{cases} -\cos \bar{\beta}_1 \cos \bar{\beta}_2 \\ -\sin \bar{\beta}_1 \cos \bar{\beta}_2 \\ 0 \end{cases}$$  (69)

have been used, see also eq. (43).

The part $\mathbf{G}_{IK}(b)$ is then defined by

$$\mathbf{G}_{IK}(b) = \begin{bmatrix} 0_{3 \times 3} & A d_1 + B d_{1, \beta} \\ d_{1}^T A + d_{1, \beta}^T C \bar{a}_\alpha d_{11} \bar{\beta}_{\gamma \beta} N_1 N_K M_{\alpha \beta} + \bar{a}_\alpha^T d_{11} D & A d_2 + B d_{2, \beta} \\ d_{2}^T A + d_{2, \beta}^T C \bar{a}_\alpha d_{21} \bar{\beta}_{\gamma \beta} N_1 N_K M_{\alpha \beta} + \bar{a}_\alpha^T d_{21} D & \bar{a}_\alpha^T d_{12} \bar{\beta}_{\gamma \beta} N_1 N_K M_{\alpha \beta} + \bar{a}_\alpha^T d_{22} D \end{bmatrix}$$  (70)

with the definitions

$$A = N_{1, \alpha} N_K \beta M_{\alpha \beta}$$

$$B = N_{1, \alpha} N_K M_{\alpha \beta}$$

$$C = N_1 N_K \alpha M_{\alpha \beta}$$

$$D = (N_{1, \beta} N_K + N_1 N_K \beta) M_{\alpha \beta}$$  (71)

The consistent formulation of the geometrical matrix is an essential condition for a quadratical convergence behaviour within the Newton–iteration.

**Remark 2:**

1) The given formulation is completely nonlinear. Thus no discussion is necessary which terms have to be used and which not as in classical geometrical nonlinear shell theories, see e.g. Pietraszkiewicz 7.

2) There is no simplification in the description of the director vector and its derivations. Thus a discussion of additional terms in $K_\sigma$, e.g. $K_{GI}$, see Ramm, Matzenmiller 11 is not necessary.

3) With the introduced director vector it is possible to describe rotations without limitations.

4) The completely nonlinear formulation is essential in the deep nonlinear range.
The residual and the tangent stiffness matrix are integrated numerically – as usual in an isoparametric formulation. A reduced integration scheme for the shear terms is a standard method to avoid shear locking.

Here, the shear part of the four node element is modified similar to the formulation of Bathe/Dvorkin\(^8,9\) which seems to be up to now the best formulation for Reissner–Mindlin based plate and shell formulations. A similar procedure for 8- and 9-node elements is proposed by Pinsky\(^29\).

For this purpose the shear strains \(\tilde{\gamma}\) are interpolated using a constant–linear interpolation based on the displacement consistent strains \(\gamma\) at sampling points, see Bathe/Dvorkin\(^8,9\), as

\[
\begin{bmatrix}
\tilde{\gamma}_{\xi 3} \\
\tilde{\gamma}_{\eta 3}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(1 - \eta)\gamma_{\xi 3B} + (1 + \eta)\gamma_{\xi 3D} \\
(1 - \xi)\gamma_{\eta 3A} + (1 + \xi)\gamma_{\eta 3C}
\end{bmatrix}.
\]

The sampling points A–D are defined in Figure 3.

![Figure 3](image)

*Figure 3* 4–node element with Bathe/Dvorkin approach

Thus the variation is given by

\[
\begin{bmatrix}
\delta\tilde{\gamma}_{\xi 3} \\
\delta\tilde{\gamma}_{\eta 3}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(1 - \eta)\delta\gamma_{\xi 3B} + (1 + \eta)\delta\gamma_{\xi 3D} \\
(1 - \xi)\delta\gamma_{\eta 3A} + (1 + \xi)\delta\gamma_{\eta 3C}
\end{bmatrix}.
\]

with

\[
\begin{align*}
\delta\gamma_{\eta 3A} &= \delta a_{\eta A} \cdot \mathbf{d}_A + a_{\eta A} \cdot \delta \mathbf{d}_A \\
\delta\gamma_{\eta 3C} &= \delta a_{\eta C} \cdot \mathbf{d}_C + a_{\eta C} \cdot \delta \mathbf{d}_C \\
\delta\gamma_{\xi 3B} &= \delta a_{\xi B} \cdot \mathbf{d}_B + a_{\xi B} \cdot \delta \mathbf{d}_B \\
\delta\gamma_{\xi 3D} &= \delta a_{\xi D} \cdot \mathbf{d}_D + a_{\xi D} \cdot \delta \mathbf{d}_D.
\end{align*}
\]

The above strains are introduced with respect to \(\xi_{\alpha} = \xi, \eta\). A transformation to \(\theta_{\beta}\) is shown in eq. (40) and bases on the introduction of the Jacobian \(J\), defined in eq. (38). It holds

\[
\gamma = J^{-1}\tilde{\gamma},
\]

\[
\begin{bmatrix}
\gamma_{13} \\
\gamma_{23}
\end{bmatrix} = J^{-1} \begin{bmatrix}
\gamma_{\xi 3} \\
\gamma_{\eta 3}
\end{bmatrix}.
\]

The principle of virtual work, see eq.(29), restricted to the shear terms is defined by

\[
D \pi_s \cdot \delta \mathbf{v} = \int_{\Omega} \delta \gamma \cdot \mathbf{Q} d\Omega = \int_{\Omega} \delta \gamma^T \mathbf{D}_s \gamma d\Omega
\]
and with eq.(75) the shear part is transformed to

\[ D \pi_s \cdot \delta \mathbf{v} = \int_\Omega \delta \tilde{\mathbf{\gamma}}^T J^{-T} D_s J^{-1} \mathbf{\gamma} d\Omega = \int_\Omega \delta \tilde{\mathbf{\gamma}}^T \tilde{D}_s \mathbf{\gamma} d\Omega = \int_\Omega \delta \tilde{\mathbf{\gamma}}^T \tilde{Q} d\Omega . \]  

(77)

Thus the formulation of the shear term can be done in the local \( \xi, \eta \) – coordinate systems.

The deformed tangential base vectors at the sampling points are defined by

\[ \bar{\mathbf{a}}_{\eta A} = \frac{1}{2}(\bar{x}_4 - \bar{x}_1) \quad \bar{\mathbf{a}}_{\xi B} = \frac{1}{2}(\bar{x}_2 - \bar{x}_1) \]

\[ \bar{\mathbf{a}}_{\eta C} = \frac{1}{2}(\bar{x}_3 - \bar{x}_2) \quad \bar{\mathbf{a}}_{\xi D} = \frac{1}{2}(\bar{x}_3 - \bar{x}_4) \]  

(78)

and the variations are

\[ \delta \bar{\mathbf{a}}_{\eta A} = \frac{1}{2}(\delta \mathbf{u}_4 - \delta \mathbf{u}_1) \quad \delta \bar{\mathbf{a}}_{\xi B} = \frac{1}{2}(\delta \mathbf{u}_2 - \delta \mathbf{u}_1) \]

\[ \delta \bar{\mathbf{a}}_{\eta C} = \frac{1}{2}(\delta \mathbf{u}_3 - \delta \mathbf{u}_2) \quad \delta \bar{\mathbf{a}}_{\xi D} = \frac{1}{2}(\delta \mathbf{u}_3 - \delta \mathbf{u}_4) . \]  

(79)

The director vector at points A–D are based on the definition, eq.(14) and the finite element approximation of the angles \( \beta_0 \) and \( \beta \), eq. (32)

\[ \bar{\beta}_A = \frac{1}{2}(\bar{\beta}_4 + \bar{\beta}_1) \quad \bar{\beta}_B = \frac{1}{2}(\bar{\beta}_1 + \bar{\beta}_2) \]

\[ \bar{\beta}_C = \frac{1}{2}(\bar{\beta}_2 + \bar{\beta}_3) \quad \bar{\beta}_D = \frac{1}{2}(\bar{\beta}_3 + \bar{\beta}_4) . \]  

(80)

For example \( \mathbf{d}_A \) is calculated from

\[ \mathbf{d}_A = \begin{bmatrix} \cos \bar{\beta}_{1A} \sin \bar{\beta}_{2A} \\ \sin \bar{\beta}_{1A} \sin \bar{\beta}_{2A} \\ \cos \bar{\beta}_{2A} \end{bmatrix} . \]  

(81)

The variation of the director vector is defined in eq. (49) and shown below again in a modified formulation

\[ \delta \mathbf{d} = \tilde{\mathbf{d}} \delta \mathbf{\beta} \]  

(82)

with

\[ \tilde{\mathbf{d}} = [\mathbf{d}_1, \mathbf{d}_2] . \]  

(83)

The variations of the \( \mathbf{\beta} \) at the sampling points are given by

\[ \delta \mathbf{\beta}_A = \frac{1}{2}(\delta \beta_4 + \delta \beta_1) \quad \delta \mathbf{\beta}_B = \frac{1}{2}(\delta \beta_1 + \delta \beta_2) \]

\[ \delta \mathbf{\beta}_C = \frac{1}{2}(\delta \beta_2 + \delta \beta_3) \quad \delta \mathbf{\beta}_D = \frac{1}{2}(\delta \beta_3 + \delta \beta_4) . \]  

(84)

For example \( \delta \mathbf{d}_A \) is then calculated from

\[ \delta \mathbf{d}_A = \tilde{\mathbf{d}}_A \delta \mathbf{\beta}_A . \]  

(85)
Thus the $B$-matrices for the shear part are given for the nodes $I = 1 - 4$ by

$$
\begin{align*}
\tilde{\mathbf{B}}_1^s &= \begin{bmatrix} d_B^T N_{1, \xi} & -\tilde{a}_{\xi B}^T d_{1B} N_{1, \xi} & -\tilde{a}_{\eta B}^T d_{2B} N_{1, \eta} \\
 d_A^T N_{1, \eta} & -\tilde{a}_{\eta A}^T d_{1A} N_{1, \eta} & -\tilde{a}_{\eta A}^T d_{2A} N_{1, \eta} \end{bmatrix} \\
\tilde{\mathbf{B}}_2^s &= \begin{bmatrix} d_B^T N_{2, \xi} & \tilde{a}_{\xi B}^T d_{1B} N_{2, \xi} & \tilde{a}_{\xi B}^T d_{2B} N_{2, \eta} \\
 d_C^T N_{2, \eta} & -\tilde{a}_{\eta C}^T d_{1C} N_{2, \eta} & -\tilde{a}_{\eta C}^T d_{2C} N_{2, \eta} \end{bmatrix} \\
\tilde{\mathbf{B}}_3^s &= \begin{bmatrix} d_D^T N_{3, \xi} & \tilde{a}_{\xi D}^T d_{1D} N_{3, \xi} & \tilde{a}_{\xi D}^T d_{2D} N_{3, \eta} \\
 d_C^T N_{3, \eta} & \tilde{a}_{\eta C}^T d_{1C} N_{3, \eta} & \tilde{a}_{\eta C}^T d_{2C} N_{3, \eta} \end{bmatrix} \\
\tilde{\mathbf{B}}_4^s &= \begin{bmatrix} d_D^T N_{4, \xi} & -\tilde{a}_{\xi D}^T d_{1D} N_{4, \xi} & -\tilde{a}_{\xi D}^T d_{2D} N_{4, \eta} \\
 d_A^T N_{4, \eta} & \tilde{a}_{\eta A}^T d_{1A} N_{4, \eta} & \tilde{a}_{\eta A}^T d_{2A} N_{4, \eta} \end{bmatrix}.
\end{align*}
$$

The residual (on element level), see eq. (58) is then modified for the shear term to

$$
G^h_{es}(\mathbf{v}_e, \delta \mathbf{v}_e) = \sum_{I=1}^{nel} \delta \mathbf{v}_I^T \int_{\Omega_e} \mathbf{\tilde{B}}_I^T (\mathbf{v}_e) \mathbf{\tilde{\Omega}}(\mathbf{v}_e) d\Omega.
$$

Similar the material part of the tangent stiffness matrix is calculated from

$$
DG^h_{es}(\mathbf{v}_e, \delta \mathbf{v}_e) \Delta \mathbf{v}_e = \sum_{I=1}^{nel} \sum_{K=1}^{nel} \delta \mathbf{v}_I^T \int_{\Omega_e} \mathbf{\tilde{B}}_{I}s(\mathbf{v}_e) \mathbf{D}s \mathbf{\tilde{B}}_{K}s(\mathbf{v}_e) d\Omega.
$$

For the geometric part of $\mathbf{K}_T$ we need the second variation of the shear strains. Based on the introduced shear strain field, eq. (72), it holds

$$
\begin{bmatrix} \delta \Delta \tilde{\gamma}_{\xi3} \\ \delta \Delta \tilde{\gamma}_{\eta3} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (1 - \eta) \delta \Delta \gamma_{\xi3B} + (1 + \eta) \delta \Delta \gamma_{\xi3D} \\ (1 - \xi) \delta \Delta \gamma_{\eta3A} + (1 + \xi) \delta \Delta \gamma_{\eta3C} \end{bmatrix},
$$

with

$$
\delta \Delta \gamma_{\eta3A} = \delta \mathbf{a}_{\eta A} \cdot \Delta \mathbf{d}_A + \Delta \mathbf{a}_{\eta A} \cdot \delta \mathbf{d}_A + \tilde{\mathbf{a}}_{\eta A} \cdot \delta \Delta \mathbf{d}_A
$$

$$
\delta \Delta \gamma_{\xi3B} = \delta \mathbf{a}_{\xi B} \cdot \Delta \mathbf{d}_B + \Delta \mathbf{a}_{\xi B} \cdot \delta \mathbf{d}_B + \tilde{\mathbf{a}}_{\xi B} \cdot \delta \Delta \mathbf{d}_B
$$

$$
\delta \Delta \gamma_{\eta3C} = \delta \mathbf{a}_{\eta C} \cdot \Delta \mathbf{d}_C + \Delta \mathbf{a}_{\eta C} \cdot \delta \mathbf{d}_C + \tilde{\mathbf{a}}_{\eta C} \cdot \delta \Delta \mathbf{d}_C
$$

$$
\delta \Delta \gamma_{\xi3D} = \delta \tilde{\mathbf{a}}_{\xi D} \cdot \Delta \mathbf{d}_D + \Delta \tilde{\mathbf{a}}_{\xi D} \cdot \delta \mathbf{d}_D + \tilde{\mathbf{a}}_{\xi D} \cdot \delta \Delta \mathbf{d}_D.
$$

The vectors in the first and second dot product have been defined in eqs. (79),(82-85). Thus only the second variation of $\mathbf{d}$ has to be defined additionally. This can be done similar to eq. (64). Based on the scalar products between the deformed base vectors and the second variation of $\mathbf{d}$, see the third terms in eq. (90) one can introduce the following matrices

$$
\begin{align*}
\mathbf{A}_A &= \begin{bmatrix} \tilde{\mathbf{a}}_{\eta A}^T d_{11A} & \tilde{\mathbf{a}}_{\eta A}^T d_{12A} \\
 \tilde{\mathbf{a}}_{\eta A}^T d_{21A} & \tilde{\mathbf{a}}_{\eta A}^T d_{22A} \end{bmatrix} & \mathbf{A}_B &= \begin{bmatrix} \tilde{\mathbf{a}}_{\xi B}^T d_{11B} & \tilde{\mathbf{a}}_{\xi B}^T d_{12B} \\
 \tilde{\mathbf{a}}_{\xi B}^T d_{21B} & \tilde{\mathbf{a}}_{\xi B}^T d_{22B} \end{bmatrix} \\
\mathbf{A}_C &= \begin{bmatrix} \tilde{\mathbf{a}}_{\eta C}^T d_{11C} & \tilde{\mathbf{a}}_{\eta C}^T d_{12C} \\
 \tilde{\mathbf{a}}_{\eta C}^T d_{21C} & \tilde{\mathbf{a}}_{\eta C}^T d_{22C} \end{bmatrix} & \mathbf{A}_D &= \begin{bmatrix} \tilde{\mathbf{a}}_{\xi D}^T d_{11D} & \tilde{\mathbf{a}}_{\xi D}^T d_{12D} \\
 \tilde{\mathbf{a}}_{\xi D}^T d_{21D} & \tilde{\mathbf{a}}_{\xi D}^T d_{22D} \end{bmatrix}.
\end{align*}
$$
After some algebra we end up with the shear part of the geometric part of the tangent stiffness matrix

\[
\tilde{\mathbf{G}}_{IK}(s) = \begin{bmatrix}
0_{3\times3} & \frac{1}{8} \mathbf{g}_{IK}
\end{bmatrix}
\]

with

\[
\begin{align*}
\mathbf{g}_{11} &= -(1 - \eta)Q_{\xi3}\dd{B} - (1 - \xi)Q_{\eta3}\dd{A} && \mathbf{g}_{11} = (1 - \eta)Q_{\xi3}\mathbf{A}_B + (1 - \xi)Q_{\eta3}\mathbf{A}_A \\
\mathbf{g}_{22} &= (1 - \eta)Q_{\xi3}\dd{B} - (1 + \xi)Q_{\eta3}\dd{C} && \mathbf{g}_{22} = (1 - \eta)Q_{\xi3}\mathbf{A}_B + (1 + \xi)Q_{\eta3}\mathbf{A}_C \\
\mathbf{g}_{33} &= (1 + \eta)Q_{\xi3}\dd{D} + (1 + \xi)Q_{\eta3}\dd{C} && \mathbf{g}_{33} = (1 + \eta)Q_{\xi3}\mathbf{A}_D + (1 + \xi)Q_{\eta3}\mathbf{A}_C \\
\mathbf{g}_{44} &= -(1 + \eta)Q_{\xi3}\dd{D} + (1 - \xi)Q_{\eta3}\dd{A} && \mathbf{g}_{44} = (1 + \eta)Q_{\xi3}\mathbf{A}_D + (1 - \xi)Q_{\eta3}\mathbf{A}_A \\
\mathbf{g}_{12} &= -(1 - \eta)Q_{\xi3}\dd{B} && \mathbf{g}_{12} = (1 - \eta)Q_{\xi3}\mathbf{A}_B \\
\mathbf{g}_{13} &= 0 && \mathbf{g}_{13} = 0 \\
\mathbf{g}_{14} &= -(1 - \xi)Q_{\eta3}\dd{A} && \mathbf{g}_{14} = (1 - \xi)Q_{\eta3}\mathbf{A}_A \\
\mathbf{g}_{23} &= -(1 + \xi)Q_{\eta3}\dd{C} && \mathbf{g}_{23} = (1 + \xi)Q_{\eta3}\mathbf{A}_C \\
\mathbf{g}_{24} &= 0 && \mathbf{g}_{24} = 0 \\
\mathbf{g}_{34} &= (1 + \eta)Q_{\xi3}\dd{D} && \mathbf{g}_{34} = (1 + \eta)Q_{\xi3}\mathbf{A}_D \\
\mathbf{g}_{KK} &= -\mathbf{g}_{IK} && \mathbf{g}_{KK} = -\mathbf{g}_{IK}
\end{align*}
\]

Thus all terms for the Bathe/Dvorkin approach have been defined. The shear terms are no longer underintegrated. Thus a 2x2 quadrature – the same as for the bending and membrane terms – is used.

**Remark 3:**

1) The formulation of the Bathe/Dvorkin approach used in this element is similar to that described in Simo et.al 1. Differences occur in all terms, describing the rotational behaviour due to the different approach of describing finite rotations. Furthermore this approach is described for the degenerated shell element of Ramm 3,4 in Stander et.al 12.

2) The formulation of the Bathe/Dvorkin approach is given here with respect to the local \(\xi_\alpha\)-system. Thus shear strains and shear forces have to be transformed to the global directions \(\theta_\alpha\) by the transformation eq. (75) for an output of these values.

**EXAMPLES**

The developed element has been implemented in an extended version of the finite element program **FEAP**. A description of this program can be found in Zienkiewicz, Taylor 10.

We want to show three types of examples. First of all we show the ability of our element to describe finite rotations in standard linear elastic test problems, then we give examples for composite shells with finite rotations and finally we discuss the application of the element to a dynamic problem.
Clamped spherical rubber shell

The nonlinear collapse behaviour of a clamped spherical shell of rubber material was analyzed by Taber \textsuperscript{13} experimentally and analytically. This solution does not take into account shear deformations. The system and material data are

\[ E = 4000 \text{ kPa} \quad \nu = 0.5 \quad R = 26.3 \text{ mm} \quad h = 4.4 \text{ mm} \]

and the finite element mesh is shown in Figure 4.

![Finite element mesh of spherical rubber shell](image)

\textit{Figure 4} Spherical rubber shell under point load

We have chosen this example to demonstrate that no problems occur in our formulation even if we have a singularity for the undeformed director at \( \mathbf{x} = \{0, 0, R\}^T \). The shell is analyzed using one quadrant with a 16x16 finite element mesh. We increase the shear correction factor \( \kappa \) by a factor of 100 to suppress the shear deformations, see Simo \textsuperscript{1}. A theoretical background for this technique is that the shear dependent term in the tangent stiffness matrix can be used as a penalty term for a Kirchhoff–type theory, see e.g. Zienkiewicz, Taylor \textsuperscript{10}.

Figure 5 show different load–deflection curves for this problem. There is an excellent agreement between experimental and computed results. The results of the present formulation differ slightly from that published by Simo \textsuperscript{1}, which may depend on the different finite rotation formulation or the different membrane part formulation. In addition the results of an axisymmetric shell element with finite rotations, see Wagner \textsuperscript{14}, are depicted in the Figure.

![Normalized load–deflection curves](image)

\textit{Figure 5} Normalized load deflection curves for a rubber shell under point load
The calculations are performed in 10 displacement steps. Applying this solution strategy the calculation of the entire load–displacement curve can be done in an efficient way. The Newton–type convergence behaviour is shown in table 1 for the norm $\|G\|_2$ of the residual for some displacement increments.

<table>
<thead>
<tr>
<th>displ.</th>
<th>norm $|G|_2$ in Iteration No.</th>
<th>load</th>
<th>F/Eh²</th>
</tr>
</thead>
<tbody>
<tr>
<td>w/R</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0.1</td>
<td>1.18E + 06</td>
<td>3.57E + 04</td>
<td>7.55E + 02</td>
</tr>
<tr>
<td>0.4</td>
<td>1.13E + 06</td>
<td>4.71E + 04</td>
<td>1.44E + 03</td>
</tr>
<tr>
<td>0.7</td>
<td>1.08E + 06</td>
<td>5.82E + 04</td>
<td>1.38E + 03</td>
</tr>
<tr>
<td>1.0</td>
<td>1.07E + 06</td>
<td>7.81E + 04</td>
<td>1.94E + 03</td>
</tr>
</tbody>
</table>

Table 1 Convergence behaviour for spherical rubber shell

The final deformed system at a deflection w=R is shown in Figure 6.

![Spherical rubber shell](image)

*Figure 6* Spherical rubber shell under point load, deformed mesh at w/R = 1

**Clamped cylindrical Composite shell**

The nonlinear behaviour of a clamped cylindrical shell panel of composite material under a uniform load for a cross ply $[0^\circ, 90^\circ]$ was analyzed by Reddy using a displacement finite element model based on the von Kármán equations. The material and geometrical data are

$$E_1 = 25 \cdot 10^6 psi \quad G_{12} = 0.5 \cdot 10^6 psi \quad R = 2540 in$$

$$E_2 = 1 \cdot 10^6 psi \quad G_{13} = 0.5 \cdot 10^6 psi \quad a = 254 in$$

$$\nu = 0.25 \quad G_{23} = 0.2 \cdot 10^6 psi \quad h = 2.54 in$$

The system is shown in Figure 7.
We run this example to test the composite formulation of our element. In this example finite rotations do not occur. The load deflection curves for the uniform load versus the center deflection are depicted in Figure 8 for 4*4, 8*8 and a 16*16 finite element mesh for one quarter of the shell.

Due to the clamped boundary conditions a certain number of elements is necessary to produce accurate results, especially in the region where we have large changes in the center displacement at nearly constant external loads. The results differ slightly from that given by Reddy 25. A further discussion is not possible because in the paper of Reddy 25 no information is given on the element formulation and the finite element meshes.

Hyperboloidal Shell under two pairs of opposite loads

There exist a number of benchmark problems for shell element formulations in the geometrical linear and nonlinear case described by finite rotations. One example is a hemispherical shell under to opposite pairs of loads, see e.g. McNeal, Harder 15. Based on this example Başar et.al. 23 define a similar problem for a hyperboloidal shell with and without a composite
material behaviour. Here we want to discuss the composite hyperboloidal shell. Due to the symmetry only one eighth of the shell has to be discretized. The shell has been analyzed for two sets of laminate schemes. Both are of cross ply type with \([0^\circ, 90^\circ, 0^\circ]\) and \([90^\circ, 0^\circ, 90^\circ]\). Here \(0^\circ\) means a fiber orientation in the circumferential direction.

The geometrical and material parameters chosen by Başar et.al. \(^{23}\) are

\[
\begin{align*}
E_1 &= 40 \cdot 10^6 & G_{12} &= 0.6 \cdot 10^6 & h &= 0.04 & R_1 &= 7.5 \\
E_2 &= 1 \cdot 10^6 & G_{13} &= 0.6 \cdot 10^6 & h_i &= 0.0133 & R_2 &= 15.0 \\
\nu &= 0.25 & G_{23} &= 0.6 \cdot 10^6 & P &= 5 & H &= 20.0
\end{align*}
\]

The radius of the hyperboloidal shell is described by \(R(x_3) = \frac{R_1}{c} \sqrt{c^2 + (x_3)^2}\) with \(c = \frac{20}{\sqrt{3}}\).

One eighth of the shell is shown in Figure 9.

Results are shown for the for points A,B,C and D of the shell, see Figure 9 for both types of layer sequences. Large displacements and finite rotations occur within the nonlinear load deflection regime. A path following scheme, see e.g. Wagner, Wriggers \(^{26}\) may be used but is not necessary. Especially for the second layer sequence the path following is advantageous due to the extremely weak behaviour of the shell for low values of the external loads. In the Figures 10 and 11 we show the load deflection curves for the sequence \([0^\circ, 90^\circ, 0^\circ]\) and a finite element mesh with \(16 \times 16\) elements. The results are in good agreement with those calculated by Başar et.al. \(^{23}\) based on a \(28 \times 28\) finite element mesh.
In the Figures 12 and 13 the load deflection curves for the sequence $[90^\circ, 0^\circ, 90^\circ]$ based on a $16 \times 16$ element mesh are depicted. Slightly differences occur in the presence of large displacements for the curves $\lambda - u_A, \lambda - u_C$. If we use a $28 \times 28$ mesh too the results nearly coincide with those of Başar et.al. 23.
In the following Figures we show the deformed meshes at load level of $\lambda = 30$. Based on the undeformed system in Figure 9 the deformed meshes are shown in the $x_1 - x_3$ and in $x_2 - x_3$ plane in the Figures 14 and 15. Furthermore the Figures 16 and 17 show perspective views of the deformed hyperboloidal shells.
Figure 14  Hyperboloidal composite shell $[0^\circ, 90^\circ, 0^\circ]$ - deformed mesh in the $x_1-x_3-$, $x_2-x_3$ plane

Figure 15  Hyperboloidal composite shell $[90^\circ, 0^\circ, 90^\circ]$ - deformed mesh in the $x_1-x_3-$, $x_2-x_3$ plane

Figure 16  Hyperboloidal composite shell $[0^\circ, 90^\circ, 0^\circ]$ - deformed mesh, perspective view

Figure 17  Hyperboloidal composite shell $[90^\circ, 0^\circ, 90^\circ]$ - deformed mesh, perspective view
Rotation of a propfan–blade

This example has been chosen to demonstrate the ability of our element to describe finite rotations without difficulties. We discuss the rotation of a propfan–blade under a time dependent axial moment.

Aircrafts with low forward speed (about less than 0.5 Mach) usually have propellers whereas for higher speeds turbojets are used. Since the rise of the fuel costs during the 1970s, the propfan has been developed. It is characterized by the large number (8–10) of low–aspect–ratio highly swept blades which are twisted along the span and curved back about the axis of rotation. A typical example is the Hamilton–Standard SR3–propfan with a diameter of 2.70 m, shown in Figure 18.

Aircrafts with propfans need up to 25 % less fuel than turbofan–powered aircrafts. The development of the propfan coincides with the progress in the development of composites. In order to limit the centrifugal forces, it is essential to have a light weight structure. Thus composites are well suited as a material for propfan–blades.

Lammering\textsuperscript{30} has analyzed the Hamilton–Standard SR3–propfan. He discusses the behaviour of blades of different materials under centrifugal forces and a constant rotational speed. Furthermore he discusses the influence of the composite layer orientation on the tip displacement of the blade and on the twisting at a certain point via optimization procedures.

Due to the symmetry of the system it is sufficient to analyze only one blade. Figure 19 shows two views of the blade. The rotation of the blade occurs around the 3–axis. For this purpose the base of the blade is connected to the axis via rigid elements. The finite element discretization is defined by 798 nodes, 740 shell elements and 3990 degrees of freedom.
The blade has a variable thickness. During the finite element discretization we use for the thickness \( h \) on element level based on the isoparametric concept \( h^h = \sum_{I=1}^{nI} N_I h_I \) with the shape functions \( N_I \) and the nodal values of the thickness \( h_I \), see eq.(31).
The distribution is shown in Figure 20.

![Figure 20 SR3-propfan: distribution of thickness in m](image)

We have chosen 6 composite layers of constant thickness (0.8 mm), three at the top and three at the bottom of the blade. Thus the distance \( \zeta_k \) between the midpoint of the considered layer and the reference surface has to be modified, see eq. (28). The chosen fiber angle sequence \([90^\circ, 45^\circ, 0^\circ, 0^\circ, 45^\circ, 90^\circ]\) is symmetric. The angles are defined with respect to axis 3, see Figure 19.

The material data are chosen as follows

\[
E_1 = 13500 \text{ kN/cm}^2, \quad E_2 = 1000 \text{ kN/cm}^2, \quad G_{12} = 540 \text{ kN/cm}^2, \quad \nu = 0.3, \quad \rho = 1600 \text{ kg/m}^3
\]

In this example we discuss the beginning of the motion of one rotor blade. The initial boundary conditions are \( \mathbf{x} = \dot{\mathbf{x}} = \ddot{\mathbf{x}} = \mathbf{0} \). The motion is initialized by an axial moment \( M_{33} = M_0 t \), with \( M_0 = 1 \text{ kN cm/sec} \) and the process time \( t \) in seconds.
A standard Newmark algorithm without damping has been used. Within the nonlinear shell element a simple row sum technique leads to a lumped mass matrix, see e.g. Hughes$^{31}$.

The results for the axis angle (angle around axis 3) versus time are depicted in Figure 21 for different time steps. Within this time the blade rotate three times.

![Figure 21 motion of the SR3–propfan, axis angle versus time](image1)

Furthermore the deformed meshes are shown for $t = 0–2.2$ seconds in steps of 0.2 secs in Figure 22 (0.0,0.2,0.4,...2.2) and for $t = 2.2–3.0$ seconds in steps of 0.2 secs in Figure 23 (2.2,2.4,2.6,2.8,3.0).

![Figure 22 motion of the SR3–propfan, $t = 0–2.2$ sec, $\Delta t = 0.2$ sec.](image2)
It can be seen that the motion is nearly a rigid body motion. The example demonstrates the ability of the element to describe finite rotations. Here we have more than 3 rotations. Thus the rotation angles are up to 1150 degrees.

CONCLUSIONS

In this paper we have derived a simple finite element formulation for geometrical nonlinear shell structures. The formulation bases on a direct introduction of the isoparametric finite element formulation into the shell equations. The element allows the occurrence of finite rotations which have been described by two independent angles. A layerwise linear elastic material model for composites has been chosen. Several numerical examples show the applicability and effectiveness of the developed element. In detail we have computed the nonlinear behaviour of a clamped spherical rubber shell under single load in the presence of finite rotations. The correct implementation of the composite material law within the finite element formulation has been shown in the second example, a cylindrical shell segment. In the third example we treat the highly nonlinear behaviour of hyperboloidal shells of composite material and different layer sequences. The last example shows the application of our shell element to a dynamic problem. Here we discuss the rotation of a propfan-blade under a time dependent axial moment. The accelerated motion has been calculated up to a rotation angle of 1150 degrees without difficulties.

Thus the presented formulation seems to be an very attractive tool for static and dynamic problems of thin composite shells in the presence of arbitrary rotations.

ACKNOWLEDGMENTS

The financial support of the Deutsche Forschungsgemeinschaft (DFG) for the second author is gratefully acknowledged. We thank R. Lammering for the finite element mesh input data of the propfan blade.
References:


2. Wagner W., Stein E. A New Finite Element Formulation for Cylindrical Shells of Composite Material, accepted for publication in Composites Engineering


5. Tsai, W. S. *Composites Design*, Think Composites, Dayton (1988)


23 Ba¸sar, Y., Ding, Y., Menzel, W., Montag, U. Finite Rotation Shell Elements via Finite–Rotation Shell Theories, Statik und Dynamik im konstruktiven Ingenieurbau, Festschrift Wilfried B. Kr"{a}tzig, 1992


List of Figures

Figure 1 Kinematic of a thin shell
Figure 2 Description of the director vector d in shell space
Figure 3 4-node element with Bathe/Dvorkin approach
Figure 4 Spherical rubber shell under point load
Figure 5 Normalized load deflection curves for a rubber shell under point load
Figure 6 Spherical rubber shell under point load, deformed mesh at w/R = 1
Figure 7 Cylindrical composite shell panel
Figure 8 Load deflection curves for a cylindrical shell segment under uniform load
Figure 9 Hyperboloidal composite shell
Figure 10 Hyperboloidal composite shell [0°, 90°, 0°], λ - u_B, λ - u_C
Figure 11 Hyperboloidal composite shell [0°, 90°, 0°], λ - u_A, λ - u_D
Figure 12 Hyperboloidal composite shell [90°, 0°, 90°], λ - u_B, λ - u_C
Figure 13 Hyperboloidal composite shell [90°, 0°, 90°], λ - u_A, λ - u_D
Figure 14 Hyperboloidal composite shell [0°, 90°, 0°] - deformed mesh in the x_1 – x_3 –, x_2 – x_3 plane
Figure 15 Hyperboloidal composite shell [90°, 0°, 90°] – deformed mesh in the x_1 – x_3 –, x_2 – x_3 plane
Figure 16 Hyperboloidal composite shell [0°, 90°, 0°] – deformed mesh, perspective view
Figure 17 Hyperboloidal composite shell [90°, 0°, 90°] – deformed mesh, perspective view
Figure 18 Hamilton–Standard SR3–propfan
Figure 19 Two views of the FE–mesh of the SR3–propfan
Figure 20 SR3–propfan: distribution of thickness in m
Figure 21 motion of the SR3–propfan, axis angle versus time
Figure 22 motion of the SR3–propfan, t = 0 – 2.2 sec, Δ t = 0.2 sec.
Figure 23 motion of the SR3–propfan, t = 2.2 – 3.0 sec, Δ t = 0.2 sec.